CSE 4th SEM

Unit-3(DAA)

BFS algorithm

In this article, we will discuss the BFS algorithm in the data structure. Breadth-first search is a graph traversal algorithm that starts traversing the graph from the root node and explores all the neighboring nodes. Then, it selects the nearest node and explores all the unexplored nodes. While using BFS for traversal, any node in the graph can be considered as the root node.

There are many ways to traverse the graph, but among them, BFS is the most commonly used approach. It is a recursive algorithm to search all the vertices of a tree or graph data structure. BFS puts every vertex of the graph into two categories - visited and non-visited. It selects a single node in a graph and, after that, visits all the nodes adjacent to the selected node.

Applications of BFS algorithm

The applications of breadth-first-algorithm are given as follows -

- BFS can be used to find the neighboring locations from a given source location.
- In a peer-to-peer network, BFS algorithm can be used as a traversal method to find all the neighboring nodes. Most torrent clients, such as BitTorrent, uTorrent, etc. employ this process to find "seeds" and "peers" in the network.
- BFS can be used in web crawlers to create web page indexes. It is one of the main algorithms that can be used to index web pages. It starts traversing from the source page and follows the links associated with the page. Here, every web page is considered as a node in the graph.
- BFS is used to determine the shortest path and minimum spanning tree.
- BFS is also used in Cheney's technique to duplicate the garbage collection.
- It can be used in ford-Fulkerson method to compute the maximum flow in a flow network.

Algorithm

The steps involved in the BFS algorithm to explore a graph are given as follows -

Step 1: SET STATUS = 1 (ready state) for each node in G

Step 2: Enqueue the starting node A and set its STATUS = 2 (waiting state)

Step 3: Repeat Steps 4 and 5 until QUEUE is empty

Step 4: Dequeue a node N. Process it and set its STATUS = 3 (processed state).

Step 5: Enqueue all the neighbours of N that are in the ready state (whose STATUS = 1) and set

their STATUS = 2

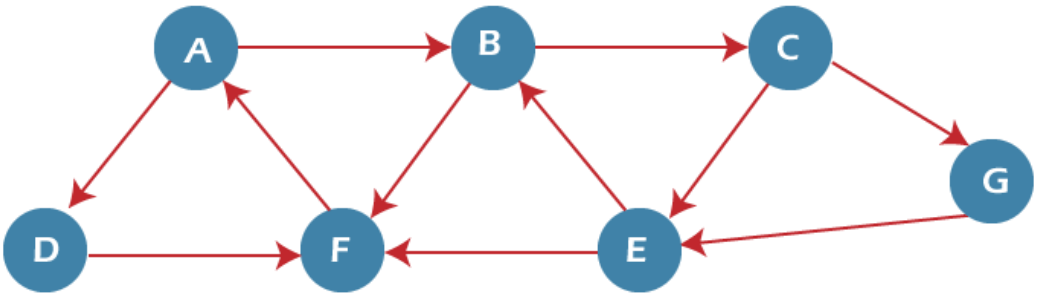
(waiting state)

[END OF LOOP]

Step 6: EXIT

Example of BFS algorithm

Now, let's understand the working of BFS algorithm by using an example. In the example given below, there is a directed graph having 7 vertices.



Adjacency Lists

A : B, D
B : C, F
C : E, G
G : E
E : B, F
F : A
D : F

In the above graph, minimum path 'P' can be found by using the BFS that will start from Node A and end at Node E. The algorithm uses two queues, namely QUEUE1 and QUEUE2. QUEUE1 holds all the nodes that are to be processed, while QUEUE2 holds all the nodes that are processed and deleted from QUEUE1.

Now, let's start examining the graph starting from Node A.

Step 1 - First, add A to queue1 and NULL to queue2.

1. QUEUE1 = {A}
2. QUEUE2 = {NULL}

Step 2 - Now, delete node A from queue1 and add it into queue2. Insert all neighbors of node A to queue1.

1. QUEUE1 = {B, D}
2. QUEUE2 = {A}

Step 3 - Now, delete node B from queue1 and add it into queue2. Insert all neighbors of node B to queue1.

1. QUEUE1 = {D, C, F}
2. QUEUE2 = {A, B}

Step 4 - Now, delete node D from queue1 and add it into queue2. Insert all neighbors of node D to queue1. The only neighbor of Node D is F since it is already inserted, so it will not be inserted again.

1. QUEUE1 = {C, F}
2. QUEUE2 = {A, B, D}

Step 5 - Delete node C from queue1 and add it into queue2. Insert all neighbors of node C to queue1.

1. QUEUE1 = {F, E, G}
2. QUEUE2 = {A, B, D, C}

Step 5 - Delete node F from queue1 and add it into queue2. Insert all neighbors of node F to queue1. Since all the neighbors of node F are already present, we will not insert them again.

1. QUEUE1 = {E, G}
2. QUEUE2 = {A, B, D, C, F}

Step 6 - Delete node E from queue1. Since all of its neighbors have already been added, so we will not insert them again. Now, all the nodes are visited, and the target node E is encountered into queue2.

1. QUEUE1 = {G}
2. QUEUE2 = {A, B, D, C, F, E}

Complexity of BFS algorithm

Time complexity of BFS depends upon the data structure used to represent the graph. The time complexity of BFS algorithm is **O(V+E)**, since in the worst case, BFS algorithm explores every node and edge. In a graph, the number of vertices is O(V), whereas the number of edges is O(E).

The space complexity of BFS can be expressed as **O(V)**, where V is the number of vertices.

DFS (Depth First Search) algorithm

In this article, we will discuss the DFS algorithm in the data structure. It is a recursive algorithm to search all the vertices of a tree data structure or a graph. The depth-first search (DFS) algorithm starts with the initial node of graph G and goes deeper until we find the goal node or the node with no children.

Because of the recursive nature, stack data structure can be used to implement the DFS algorithm. The process of implementing the DFS is similar to the BFS algorithm.

The step by step process to implement the DFS traversal is given as follows -

1. First, create a stack with the total number of vertices in the graph.
2. Now, choose any vertex as the starting point of traversal, and push that vertex into the stack.
3. After that, push a non-visited vertex (adjacent to the vertex on the top of the stack) to the top of the stack.
4. Now, repeat steps 3 and 4 until no vertices are left to visit from the vertex on the stack's top.
5. If no vertex is left, go back and pop a vertex from the stack.
6. Repeat steps 2, 3, and 4 until the stack is empty.

Applications of DFS algorithm

The applications of using the DFS algorithm are given as follows -

- DFS algorithm can be used to implement the topological sorting.
- It can be used to find the paths between two vertices.
- It can also be used to detect cycles in the graph.
- DFS algorithm is also used for one solution puzzles.
- DFS is used to determine if a graph is bipartite or not.

Algorithm

Step 1: SET STATUS = 1 (ready state) for each node in G

Step 2: Push the starting node A on the stack and set its STATUS = 2 (waiting state)

Step 3: Repeat Steps 4 and 5 until STACK is empty

Step 4: Pop the top node N. Process it and set its STATUS = 3 (processed state)

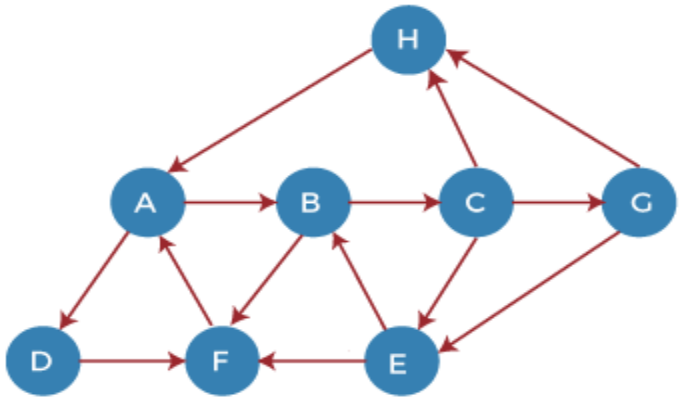
Step 5: Push on the stack all the neighbors of N that are in the ready state (whose STATUS = 1) and set their STATUS = 2 (waiting state)

[END OF LOOP]

Step 6: EXIT

Example of DFS algorithm

Now, let's understand the working of the DFS algorithm by using an example. In the example given below, there is a directed graph having 7 vertices.



Adjacency Lists

A : B, D
B : C, F
C : E, G, H
G : E, H
E : B, F
F : A
D : F
H : A

Now, let's start examining the graph starting from Node H.

Step 1 - First, push H onto the stack.

1. STACK: H

Step 2 - POP the top element from the stack, i.e., H, and print it. Now, PUSH all the neighbors of H onto the stack that are in ready state.

1. Print: H]STACK: A

Step 3 - POP the top element from the stack, i.e., A, and print it. Now, PUSH all the neighbors of A onto the stack that are in ready state.

1. Print: A
2. STACK: B, D

Step 4 - POP the top element from the stack, i.e., D, and print it. Now, PUSH all the neighbors of D onto the stack that are in ready state.

1. Print: D
2. STACK: B, F

Step 5 - POP the top element from the stack, i.e., F, and print it. Now, PUSH all the neighbors of F onto the stack that are in ready state.

1. Print: F
2. STACK: B

Step 6 - POP the top element from the stack, i.e., B, and print it. Now, PUSH all the neighbors of B onto the stack that are in ready state.

1. Print: B
2. STACK: C

Step 7 - POP the top element from the stack, i.e., C, and print it. Now, PUSH all the neighbors of C onto the stack that are in ready state.

1. Print: C
2. STACK: E, G

Step 8 - POP the top element from the stack, i.e., G and PUSH all the neighbors of G onto the stack that are in ready state.

1. Print: G
2. STACK: E

Step 9 - POP the top element from the stack, i.e., E and PUSH all the neighbors of E onto the stack that are in ready state.

1. Print: E
2. STACK:

Now, all the graph nodes have been traversed, and the stack is empty.

Complexity of Depth-first search algorithm

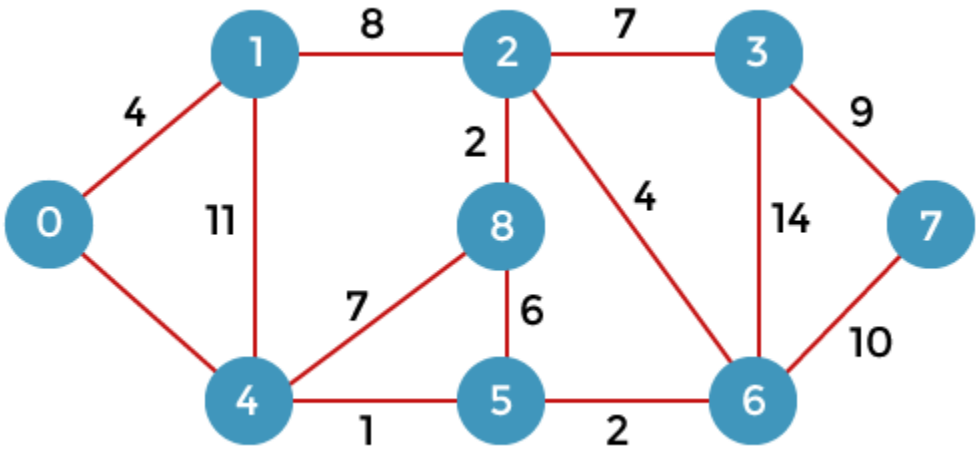
The time complexity of the DFS algorithm is **O(V+E)**, where V is the number of vertices and E is the number of edges in the graph.

The space complexity of the DFS algorithm is O(V).

Dijkstra Algorithm

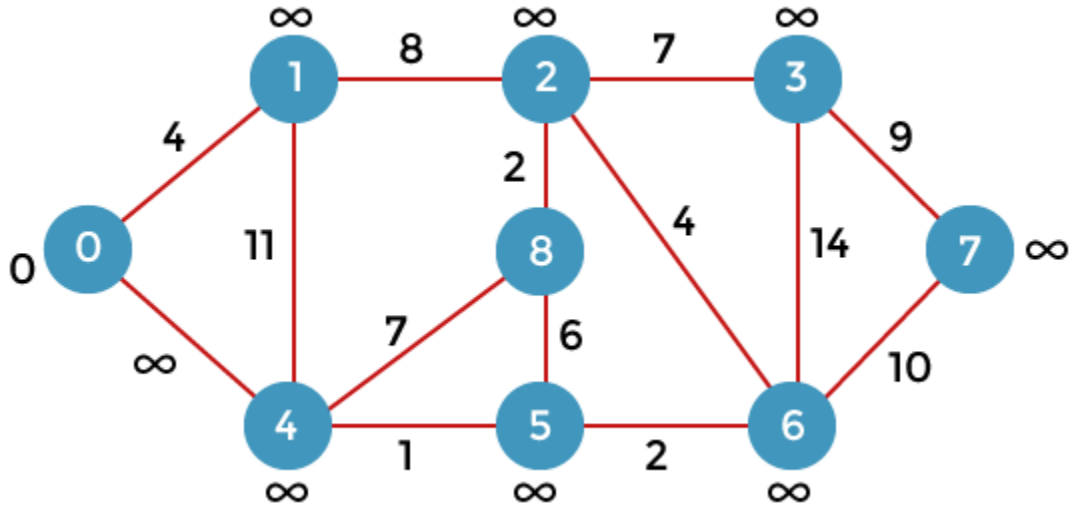
Dijkstra algorithm is a single-source shortest path algorithm. Here, single-source means that only one source is given, and we have to find the shortest path from the source to all the nodes.

Let's understand the working of Dijkstra's algorithm. Consider the below graph.



First, we have to consider any vertex as a source vertex. Suppose we consider vertex 0 as a source vertex.

Here we assume that 0 as a source vertex, and distance to all the other vertices is infinity. Initially, we do not know the distances. First, we will find out the vertices which are directly connected to the vertex 0. As we can observe in the above graph that two vertices are directly connected to vertex 0.



Let's assume that the vertex 0 is represented by 'x' and the vertex 1 is represented by 'y'. The distance between the vertices can be calculated by using the below formula:

d(x, y) = d(x) + c(x, y) < d(y)

= **(0 + 4)** < ∞

= **4** < ∞

Since 4<∞ so we will update d(v) from ∞ to 4.

Therefore, we come to the conclusion that the formula for calculating the distance between the vertices:

{if(d(u) + c(u, v) < d(v))

d(v) = d(u) +c(u, v) }

Now we consider vertex 0 same as 'x' and vertex 4 as 'y'.

d(x, y) = d(x) + c(x, y) < d(y)

= **(0 + 8)** < ∞

= **8** < ∞

Therefore, the value of d(y) is 8. We replace the infinity value of vertices 1 and 4 with the values 4 and 8 respectively. Now, we have found the shortest path from the vertex 0 to 1 and 0 to 4. Therefore, vertex 0 is selected. Now, we will compare all the vertices except the vertex 0. Since vertex 1 has the lowest value, i.e., 4; therefore, vertex 1 is selected.

Since vertex 1 is selected, so we consider the path from 1 to 2, and 1 to 4. We will not consider the path from 1 to 0 as the vertex 0 is already selected.

First, we calculate the distance between the vertex 1 and 2. Consider the vertex 1 as 'x', and the vertex 2 as 'y'.

d(x, y) = d(x) + c(x, y) < d(y)

= **(4 + 8)** < ∞

= **12** < ∞

Since 12<∞ so we will update d(2) from ∞ to 12.

Now, we calculate the distance between the vertex 1 and vertex 4. Consider the vertex 1 as 'x' and the vertex 4 as 'y'.

d(x, y) = d(x) + c(x, y) < d(y)

= (4 + 11) < 8

= 15 < 8

Since 15 is not less than 8, we will not update the value d(4) from 8 to 12.

Till now, two nodes have been selected, i.e., 0 and 1. Now we have to compare the nodes except the node 0 and 1. The node 4 has the minimum distance, i.e., 8. Therefore, vertex 4 is selected.

Since vertex 4 is selected, so we will consider all the direct paths from the vertex 4. The direct paths from vertex 4 are 4 to 0, 4 to 1, 4 to 8, and 4 to 5. Since the vertices 0 and 1 have already been selected so we will not consider the vertices 0 and 1. We will consider only two vertices, i.e., 8 and 5.

First, we consider the vertex 8. First, we calculate the distance between the vertex 4 and 8. Consider the vertex 4 as 'x', and the vertex 8 as 'y'.

d(x, y) = d(x) + c(x, y) < d(y)

= (8 + 7) < ∞

= **15** < ∞

Since 15 is less than the infinity so we update d(8) from infinity to 15.

Now, we consider the vertex 5. First, we calculate the distance between the vertex 4 and 5. Consider the vertex 4 as 'x', and the vertex 5 as 'y'.

d(x, y) = d(x) + c(x, y) < d(y)

= **(8 + 1)** < ∞

= **9** < ∞

Since 5 is less than the infinity, we update d(5) from infinity to 9.

Till now, three nodes have been selected, i.e., 0, 1, and 4. Now we have to compare the nodes except the nodes 0, 1 and 4. The node 5 has the minimum value, i.e., 9. Therefore, vertex 5 is selected.

Since the vertex 5 is selected, so we will consider all the direct paths from vertex 5. The direct paths from vertex 5 are 5 to 8, and 5 to 6.

First, we consider the vertex 8. First, we calculate the distance between the vertex 5 and 8. Consider the vertex 5 as 'x', and the vertex 8 as 'y'.

d(x, y) = d(x) + c(x, y) < d(y)

= (9 + 15) < 15

= 24 < 15

Since 24 is not less than 15 so we will not update the value d(8) from 15 to 24.

Now, we consider the vertex 6. First, we calculate the distance between the vertex 5 and 6. Consider the vertex 5 as 'x', and the vertex 6 as 'y'.

d(x, y) = d(x) + c(x, y) < d(y)

= **(9 + 2)** < ∞

= **11** < ∞

Since 11 is less than infinity, we update d(6) from infinity to 11.

Till now, nodes 0, 1, 4 and 5 have been selected. We will compare the nodes except the selected nodes. The node 6 has the lowest value as compared to other nodes. Therefore, vertex 6 is selected.

Since vertex 6 is selected, we consider all the direct paths from vertex 6. The direct paths from vertex 6 are 6 to 2, 6 to 3, and 6 to 7.

First, we consider the vertex 2. Consider the vertex 6 as 'x', and the vertex 2 as 'y'.

d(x, y) = d(x) + c(x, y) < d(y)

= (11 + 4) < 12

= 15 < 12

Since 15 is not less than 12, we will not update d(2) from 12 to 15

Now we consider the vertex 3. Consider the vertex 6 as 'x', and the vertex 3 as 'y'.

d(x, y) = d(x) + c(x, y) < d(y)

= (11 + 14) < ∞

= 25 < ∞

Since 25 is less than ∞, so we will update d(3) from ∞ to 25.

Now we consider the vertex 7. Consider the vertex 6 as 'x', and the vertex 7 as 'y'.

d(x, y) = d(x) + c(x, y) < d(y)

= (11 + 10) < ∞

= 22 < ∞

Since 22 is less than ∞ so, we will update d(7) from ∞ to 22.

Till now, nodes 0, 1, 4, 5, and 6 have been selected. Now we have to compare all the unvisited nodes, i.e., 2, 3, 7, and 8. Since node 2 has the minimum value, i.e., 12 among all the other unvisited nodes. Therefore, node 2 is selected.

Since node 2 is selected, so we consider all the direct paths from node 2. The direct paths from node 2 are 2 to 8, 2 to 6, and 2 to 3.

First, we consider the vertex 8. Consider the vertex 2 as 'x' and 8 as 'y'.

d(x, y) = d(x) + c(x, y) < d(y)

= (12 + 2) < 15

= 14 < 15

Since 14 is less than 15, we will update d(8) from 15 to 14.

Now, we consider the vertex 6. Consider the vertex 2 as 'x' and 6 as 'y'.

d(x, y) = d(x) + c(x, y) < d(y)

= (12 + 4) < 11

= 16 < 11

Since 16 is not less than 11 so we will not update d(6) from 11 to 16.

Now, we consider the vertex 3. Consider the vertex 2 as 'x' and 3 as 'y'.

d(x, y) = d(x) + c(x, y) < d(y)

= (12 + 7) < 25

= 19 < 25

Since 19 is less than 25, we will update d(3) from 25 to 19.

Till now, nodes 0, 1, 2, 4, 5, and 6 have been selected. We compare all the unvisited nodes, i.e., 3, 7, and 8. Among nodes 3, 7, and 8, node 8 has the minimum value. The nodes which are directly connected to node 8 are 2, 4, and 5. Since all the directly connected nodes are selected so we will not consider any node for the updation.

The unvisited nodes are 3 and 7. Among the nodes 3 and 7, node 3 has the minimum value, i.e., 19. Therefore, the node 3 is selected. The nodes which are directly connected to the node 3 are 2, 6, and 7. Since the nodes 2 and 6 have been selected so we will consider these two nodes.

Now, we consider the vertex 7. Consider the vertex 3 as 'x' and 7 as 'y'.

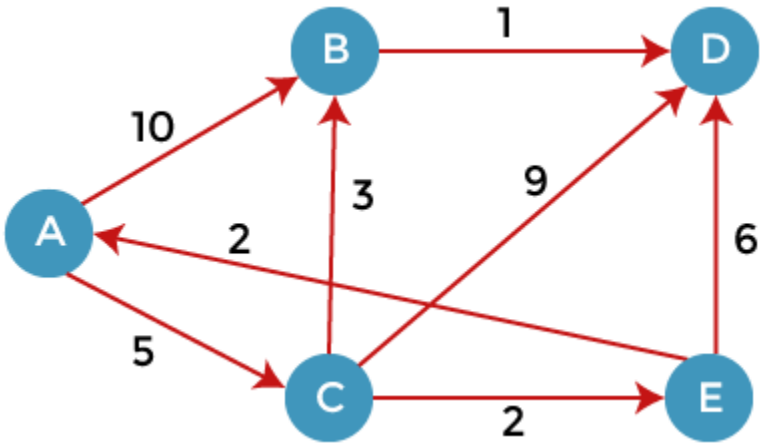
$d(x, y) = d(x) + c(x, y) < d(y)$

$= (19 + 9) < 21$

$= 28 < 21$

Since 28 is not less than 21, so we will not update d(7) from 28 to 21.

Let's consider the directed graph.



Here, we consider A as a source vertex. A vertex is a source vertex so entry is filled with 0 while other vertices filled with ∞. The distance from source vertex to source vertex is 0, and the distance from the source vertex to other vertices is ∞.

We will solve this problem using the below table:

A	B	C	D	E
∞	∞	∞	∞	∞

Since 0 is the minimum value in the above table, so we select vertex A and added in the second row shown as below:

	A	B	C	D	E
A	0	∞	∞	∞	∞

As we can observe in the above graph that there are two vertices directly connected to the vertex A, i.e., B and C. The vertex A is not directly connected to the vertex E, i.e., the edge is from E to A. Here we can calculate the two distances, i.e., from A to B and A to C. The same formula will be used as in the previous problem.

1. If(d(x) + c(x, y) **< d**(y))
2. Then we update d(y) = d(x) + c(x, y)

	A	B	C	D	E
A	0	∞	∞	∞	∞
		10	5	∞	∞

As we can observe in the third row that 5 is the lowest value so vertex C will be added in the third row.

We have calculated the distance of vertices B and C from A. Now we will compare the vertices to find the vertex with the lowest value. Since the vertex C has the minimum value, i.e., 5 so vertex C will be selected.

Since the vertex C is selected, so we consider all the direct paths from the vertex C. The direct paths from the vertex C are C to B, C to D, and C to E.

First, we consider the vertex B. We calculate the distance from C to B. Consider vertex C as 'x' and vertex B as 'y'.

d(x, y) = d(x) + c(x, y) < d(y)

= (5 + 3) < ∞

= 8 < ∞

Since 8 is less than the infinity so we update d(B) from ∞ to 8. Now the new row will be inserted in which value 8 will be added under the B column.

	A	B	C	D	E
A	0	∞	∞	∞	∞
		10	5	∞	∞
		8			

We consider the vertex D. We calculate the distance from C to D. Consider vertex C as 'x' and vertex D as 'y'.

d(x, y) = d(x) + c(x, y) < d(y)

= (5 + 9) < ∞

= 14 < ∞

Since 14 is less than the infinity so we update d(D) from ∞ to 14. The value 14 will be added under the D column.

	A	B	C	D	E
A	0	∞	∞	∞	∞
C		10	5	∞	∞
		8		14	

We consider the vertex E. We calculate the distance from C to E. Consider vertex C as 'x' and vertex E as 'y'.

d(x, y) = d(x) + c(x, y) < d(y)

= (5 + 2) < ∞

= 7 < ∞

Since 14 is less than the infinity so we update d(D) from ∞ to 14. The value 14 will be added under the D column.

	A	B	C	D	E
A	0	∞	∞	∞	∞
C		10	5	∞	∞
		8		14	7

As we can observe in the above table that 7 is the minimum value among 8, 14, and 7. Therefore, the vertex E is added on the left as shown in the below table:

	A	B	C	D	E
A	0	∞	∞	∞	∞
C		10	5	∞	∞
E		8		14	7

The vertex E is selected so we consider all the direct paths from the vertex E. The direct paths from the vertex E are E to A and E to D. Since the vertex A is selected, so we will not consider the path from E to A.

Consider the path from E to D.

d(x, y) = d(x) + c(x, y) < d(y)

= (7 + 6) < 14

= 13 < 14

Since 13 is less than the infinity so we update d(D) from ∞ to 13. The value 13 will be added under the D column.

	A	B	C	D	E
A	0	∞	∞	∞	∞
C		10	5	∞	∞
E		8		14	7
B		8		13	

The value 8 is minimum among 8 and 13. Therefore, vertex B is selected. The direct path from B is B to D.

d(x, y) = d(x) + c(x, y) < d(y)

= (8 + 1) < 13

= 9 < 13

Since 9 is less than 13 so we update d(D) from 13 to 9. The value 9 will be added under the D column.

	A	B	C	D	E
A	0	∞	∞	∞	∞
C		10	5	∞	∞
E		8		14	7
B		8		13	
D				9	

Dijkstra's Algorithm Complexity

Time Complexity: O(E Log V)

where, E is the number of edges and V is the number of vertices.

Space Complexity: O(V)

Bellman Ford's Algorithm

Bellman Ford algorithm helps us find the shortest path from a vertex to all other vertices of a weighted graph.

It is similar to Dijkstra's algorithm but it can work with graphs in which edges can have negative weights.

Why would one ever have edges with negative weights in real life?

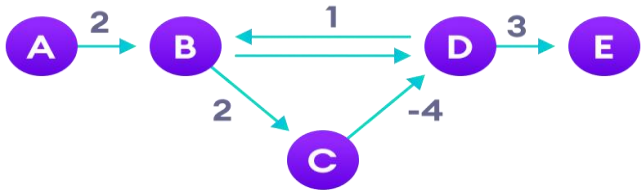
Negative weight edges might seem useless at first but they can explain a lot of phenomena like cashflow, the heat released/absorbed in a chemical reaction, etc.

For instance, if there are different ways to reach from one chemical A to another chemical B, each method will have sub-reactions involving both heat dissipation and absorption.

If we want to find the set of reactions where minimum energy is required, then we will need to be able to factor in the heat absorption as negative weights and heat dissipation as positive weights.

Why do we need to be careful with negative weights?

Negative weight edges can create negative weight cycles i.e. a cycle that will reduce the total path distance by coming back to the same point.



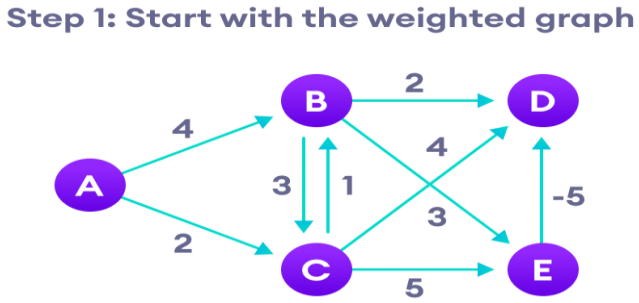
Negative weight cycles can give an incorrect result when trying to find out the shortest path

Shortest path algorithms like Dijkstra's Algorithm that aren't able to detect such a cycle can give an incorrect result because they can go through a negative weight cycle and reduce the path length.

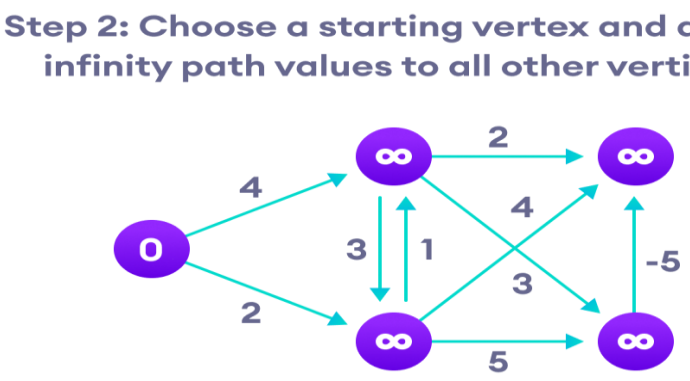
How Bellman Ford's algorithm works

Bellman Ford algorithm works by overestimating the length of the path from the starting vertex to all other vertices. Then it iteratively relaxes those estimates by finding new paths that are shorter than the previously overestimated paths.

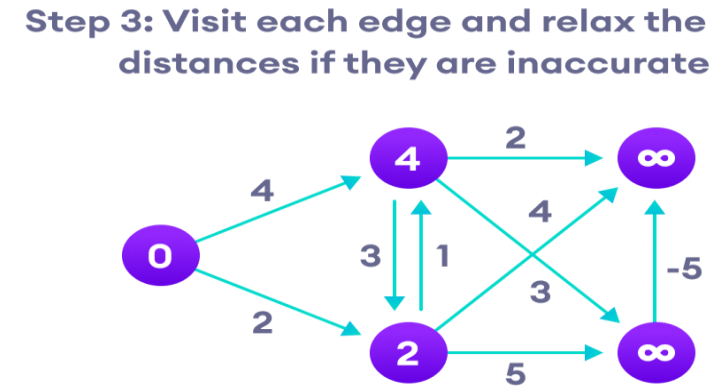
By doing this repeatedly for all vertices, we can guarantee that the result is optimized.



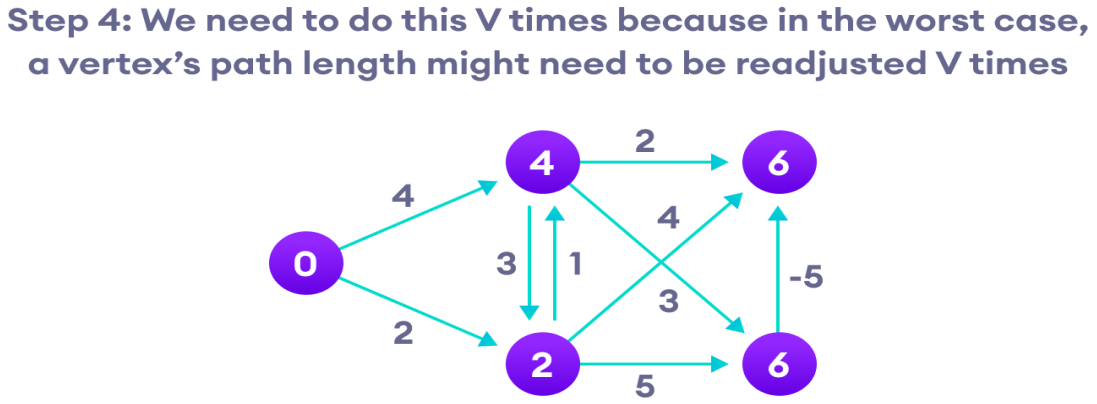
Step-1 for Bellman Ford's algorithm



Step-2 for Bellman Ford's algorithm

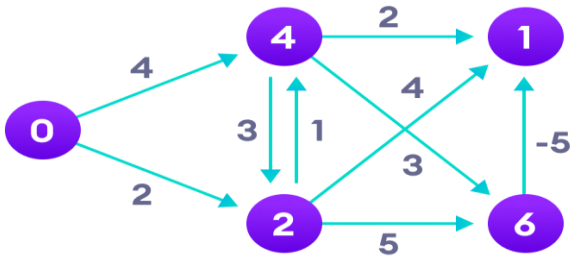


Step-3 for Bellman Ford's algorithm



Step-4 for Bellman Ford's algorithm

Step 5: Notice how the vertex at the top right corner had its path length adjusted



Step-5 for Bellman Ford's algorithm

Step 6: After all the vertices have their path lengths, we check if a negative cycle is present

	B	C	D	E
0	∞	∞	∞	∞
0	4	2	∞	∞
0	3	2	6	6
0	3	2	1	6
0	3	2	1	6

Step-6 for Bellman Ford's algorithm

Bellman Ford's Complexity

Time Complexity

Best Case Complexity	O(E)
Average Case Complexity	O(VE)
Worst Case Complexity	O(VE)

Space Complexity

And, the space complexity is O(V).

Floyd-Warshall Algorithm

In this tutorial, you will learn how floyd-warshall algorithm works. Also, you will find working examples of floyd-warshall algorithm in C, C++, Java and Python.

Floyd-Warshall Algorithm is an algorithm for finding the shortest path between all the pairs of vertices in a weighted graph. This algorithm works for both the directed and undirected weighted graphs. But, it does not work for the graphs with negative cycles (where the sum of the edges in a cycle is negative).

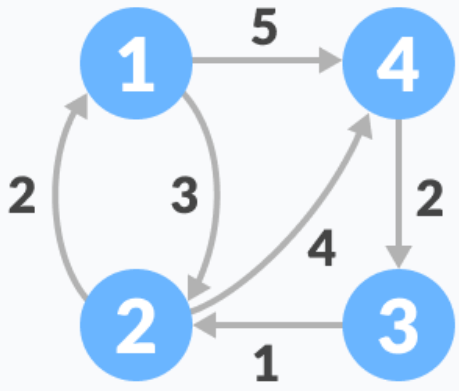
A weighted graph is a graph in which each edge has a numerical value associated with it.

Floyd-Warhshall algorithm is also called as Floyd's algorithm, Roy-Floyd algorithm, Roy-Warshall algorithm, or WFI algorithm.

This algorithm follows the dynamic programming approach to find the shortest paths.

How Floyd-Warshall Algorithm Works?

Let the given graph be:



Initial graph

Follow the steps below to find the shortest path between all the pairs of vertices.

1. Create a matrix A^0 of dimension n*n where n is the number of vertices. The row and the column are indexed as i and j respectively. i and j are the vertices of the graph.

   Each cell A[i][j] is filled with the distance from the i^{th} vertex to the j^{th} vertex. If there is no path from i^{th} vertex to j^{th} vertex, the cell is left as infinity.

$$A^0 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 3 & \infty & 5 \\ 2 & 0 & \infty & 4 \\ \infty & 1 & 0 & \infty \\ \infty & \infty & 2 & 0 \end{bmatrix} \end{array}$$

   Fill each cell with the distance between ith and jth vertex

2. Now, create a matrix A^1 using matrix A^0. The elements in the first column and the first row are left as they are. The remaining cells are filled in the following way.

   Let k be the intermediate vertex in the shortest path from source to destination. In this step, k is the first vertex. A[i][j] is filled with (A[i][k] + A[k][j]) if (A[i][j] > A[i][k] + A[k][j]).

   That is, if the direct distance from the source to the destination is greater than the path through the vertex k, then the cell is filled with A[i][k] + A[k][j].

   In this step, k is vertex 1. We calculate the distance from source vertex to destination vertex through this vertex k.

$$A^1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 3 & \infty & 5 \\ 2 & 0 & & \\ \infty & & 0 & \\ \infty & & & 0 \end{bmatrix} \end{array} \rightarrow \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 3 & \infty & 5 \\ 2 & 0 & 9 & 4 \\ \infty & 1 & 0 & 8 \\ \infty & \infty & 2 & 0 \end{bmatrix} \end{array}$$

Calculate the distance from the source vertex to destination vertex through this vertex k
For example: For $A^1[2, 4]$, the direct distance from vertex 2 to 4 is 4 and the sum of the distance from vertex 2 to 4 through vertex (ie. from vertex 2 to 1 and from vertex 1 to 4) is 7. Since $4 < 7$, $A^0[2, 4]$ is filled with 4.

3. Similarly, A^2 is created using A^1. The elements in the second column and the second row are left as they are.

   In this step, k is the second vertex (i.e. vertex 2). The remaining steps are the same as in **step 2**.

$$A^2 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 3 & & \\ 2 & 0 & 9 & 4 \\ & 1 & 0 & \\ & \infty & & 0 \end{bmatrix} \end{array} \rightarrow \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 3 & 9 & 5 \\ 2 & 0 & 9 & 4 \\ 3 & 1 & 0 & 5 \\ \infty & \infty & 2 & 0 \end{bmatrix} \end{array}$$

Calculate the distance from the source vertex to destination vertex through this vertex 2

4. Similarly, A^3 and A^4 is also created.

$$A^3 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & & \infty & \\ & 0 & 9 & \\ \infty & 1 & 0 & 8 \\ & & 2 & 0 \end{bmatrix} \end{array} \rightarrow \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 3 & 9 & 5 \\ 2 & 0 & 9 & 4 \\ 3 & 1 & 0 & 5 \\ 5 & 3 & 2 & 0 \end{bmatrix} \end{array}$$

Calculate the distance from the source vertex to destination vertex through this vertex

$$A^4 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & & & 5 \\ & 0 & & 4 \\ & & 0 & 5 \\ 5 & 3 & 2 & 0 \end{bmatrix} \end{array} \rightarrow \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 3 & 7 & 5 \\ 2 & 0 & 6 & 4 \\ 3 & 1 & 0 & 5 \\ 5 & 3 & 2 & 0 \end{bmatrix} \end{array}$$

3

Calculate the distance from the source vertex to destination vertex through this vertex 4

5. A^4 gives the shortest path between each pair of vertices.

Floyd Warshall Algorithm Complexity

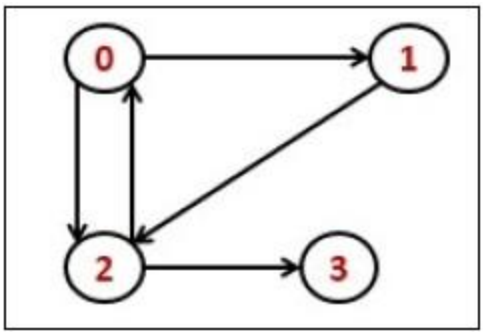
Time Complexity

There are three loops. Each loop has constant complexities. So, the time complexity of the Floyd-Warshall algorithm is $O(n^3)$.

Space Complexity

The space complexity of the Floyd-Warshall algorithm is $O(n^2)$.

Transitive Closure it the reachability matrix to reach from vertex u to vertex v of a graph. One graph is given, we have to find a vertex v which is reachable from another vertex u, for all vertex pairs (u, v).



The final matrix is the Boolean type. When there is a value 1 for vertex u to vertex v, it means that there is at least one path from u to v.

Input and Output

Input:

1 1 0 1

0 1 1 0

0 0 1 1

0 0 0 1

Output:

The matrix of transitive closure

1 1 1 1

0 1 1 1

0 0 1 1

0 0 0 1

Algorithm

transColsure(graph)

Input: The given graph.
Output: Transitive Closure matrix.

Begin

copy the adjacency matrix into another matrix named transMat

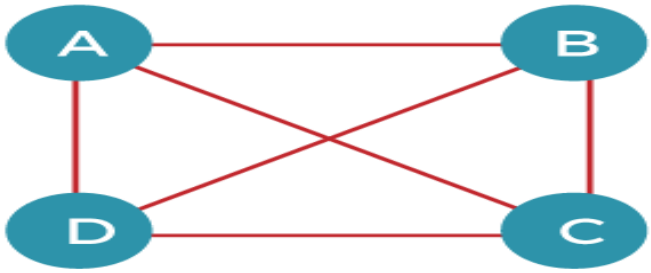
for any vertex k in the graph, do

```
    for each vertex i in the graph, do
      for each vertex j in the graph, do
        transMat[i, j] := transMat[i, j] OR (transMat[i, k]) AND transMat[k, j])
      done
    done
  done
  Display the transMat
End
```

Minimum Spanning Tree

Before knowing about the minimum spanning tree, we should know about the spanning tree.

To understand the concept of spanning tree, consider the below graph:



The above graph can be represented as G(V, E), where 'V' is the number of vertices, and 'E' is the number of edges. The spanning tree of the above graph would be represented as G`(V`, E`). In this case, V` = V means that the number of vertices in the spanning tree would be the same as the number of vertices in the graph, but the number of edges would be different. The number of edges in the spanning tree is the subset of the number of edges in the original graph. Therefore, the number of edges can be written as:

E` € E

It can also be written as:

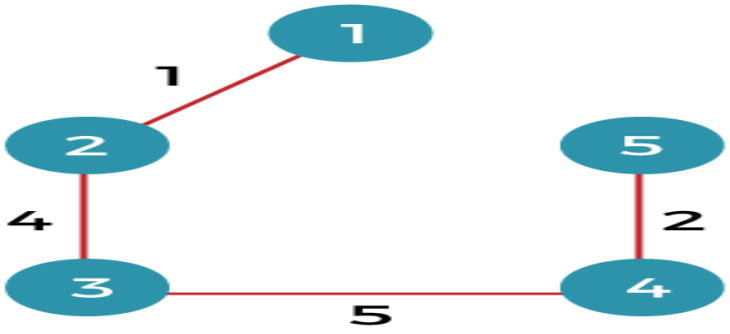
E` = |V| - 1

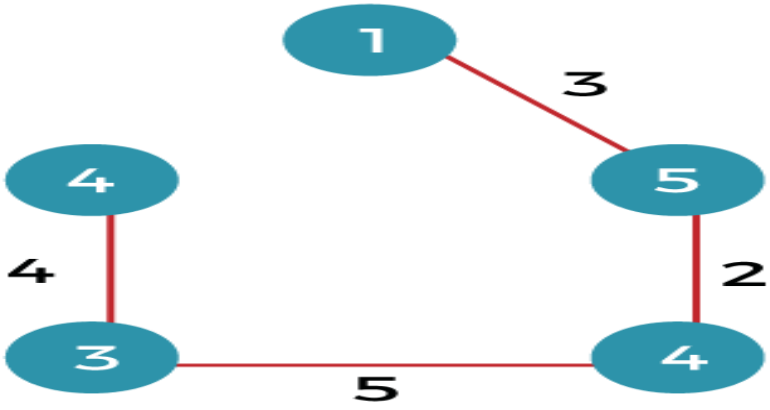
Two conditions exist in the spanning tree, which is as follows:

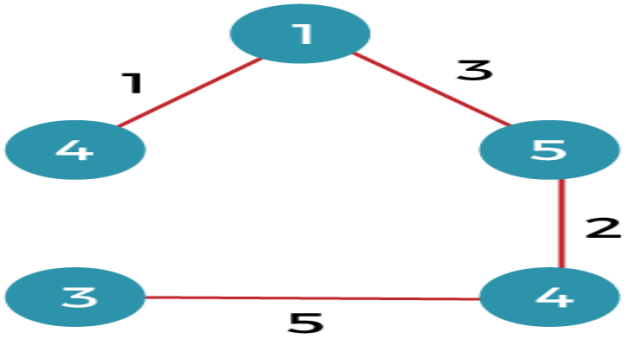
- The number of vertices in the spanning tree would be the same as the number of vertices in the original graph.
  V` = V
- The number of edges in the spanning tree would be equal to the number of edges minus 1.
  E` = |V| - 1
- The spanning tree should not contain any cycle.
- The spanning tree should not be disconnected.

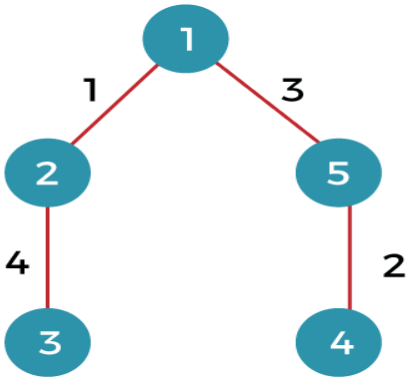
Consider the below graph:

The above graph contains 5 vertices. As we know, the vertices in the spanning tree would be the same as the graph; therefore, V` is equal 5. The number of edges in the spanning tree would be equal to (5 - 1), i.e., 4. The following are the possible spanning trees:









What is a minimum spanning tree?

The minimum spanning tree is a spanning tree whose sum of the edges is minimum. Consider the below graph that contains the edge weight:

The following are the spanning trees that we can make from the above graph.

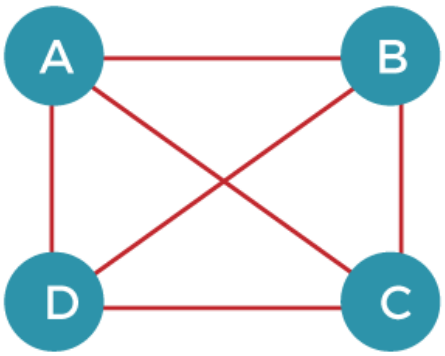
- The first spanning tree is a tree in which we have removed the edge between the vertices 1 and 5 shown as below:
  The sum of the edges of the above tree is (1 + 4 + 5 + 2): 12
- The second spanning tree is a tree in which we have removed the edge between the vertices 1 and 2 shown as below:
  The sum of the edges of the above tree is (3 + 2 + 5 + 4) : 14
- The third spanning tree is a tree in which we have removed the edge between the vertices 2 and 3 shown as below:
  The sum of the edges of the above tree is (1 + 3 + 2 + 5) : 11
- The fourth spanning tree is a tree in which we have removed the edge between the vertices 3 and 4 shown as below:
  The sum of the edges of the above tree is (1 + 3 + 2 + 4) : 10. The edge cost 10 is minimum so it is a minimum spanning tree.

General properties of minimum spanning tree:

- If we remove any edge from the spanning tree, then it becomes disconnected. Therefore, we cannot remove any edge from the spanning tree.
- If we add an edge to the spanning tree then it creates a loop. Therefore, we cannot add any edge to the spanning tree.
- In a graph, each edge has a distinct weight, then there exists only a single and unique minimum spanning tree. If the edge weight is not distinct, then there can be more than one minimum spanning tree.
- A complete undirected graph can have an n^{n-2} number of spanning trees.
- Every connected and undirected graph contains atleast one spanning tree.
- The disconnected graph does not have any spanning tree.
- In a complete graph, we can remove maximum (e-n+1) edges to construct a spanning tree.

Let's understand the last property through an example.

Consider the complete graph which is given below:



The number of spanning trees that can be made from the above complete graph equals to $n^{n-2} = 4^{4-2} = 16$.

Therefore, 16 spanning trees can be created from the above graph.

The maximum number of edges that can be removed to construct a spanning tree equals to e-n+1 = 6 - 4 + 1 = 3.

Flow Networks and Flows

Flow Network is a directed graph that is used for modeling material Flow. There are two different vertices; one is a **source** which produces material at some steady rate, and another one is sink which consumes the content at the same constant speed. The flow of the material at any mark in the system is the rate at which the element moves.

Some real-life problems like the flow of liquids through pipes, the current through wires and delivery of goods can be modeled using flow networks.

Definition: A Flow Network is a directed graph G = (V, E) such that

1. For each edge $(u, v) \in E$, we associate a nonnegative weight capacity $c(u, v) \geq 0$.If $(u, v) \notin E$, we assume that $c(u, v) = 0$.
2. There are two distinguishing points, the source s, and the sink t;
3. For every vertex $v \in V$, there is a path from s to t containing v.

Let G = (V, E) be a flow network. Let s be the source of the network, and let t be the sink. A flow in G is a real-valued function $f: V \times V \rightarrow R$ such that the following properties hold:

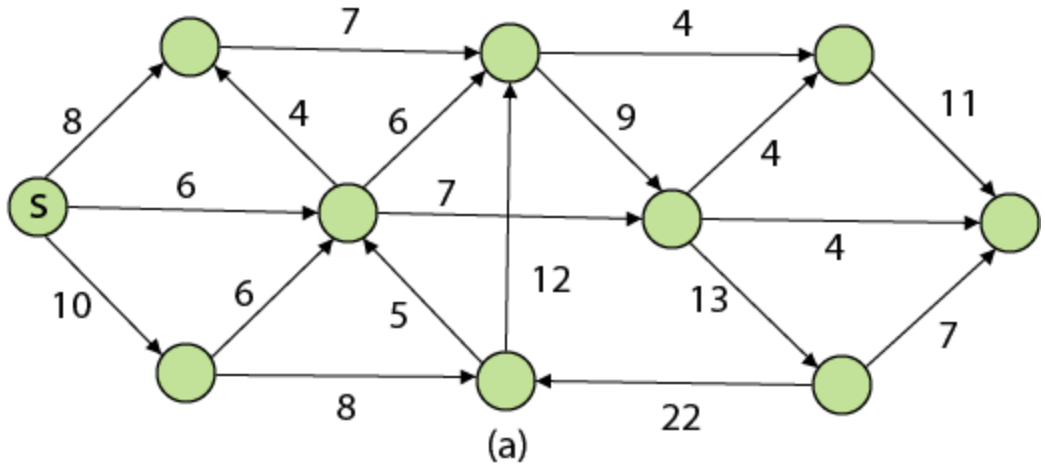
- **Capacity Constraint:** For all u, v ∈ V, we need f (u, v) ≤ c (u, v).
- **Skew Symmetry:** For all u, v ∈ V, we need f (u, v) = - f (u, v).
- **Flow Conservation:** For all u ∈ V-{s, t}, we need

$$\sum_{v \in V} f(u, v) = \sum_{u \in V} f(u, v) = 0$$

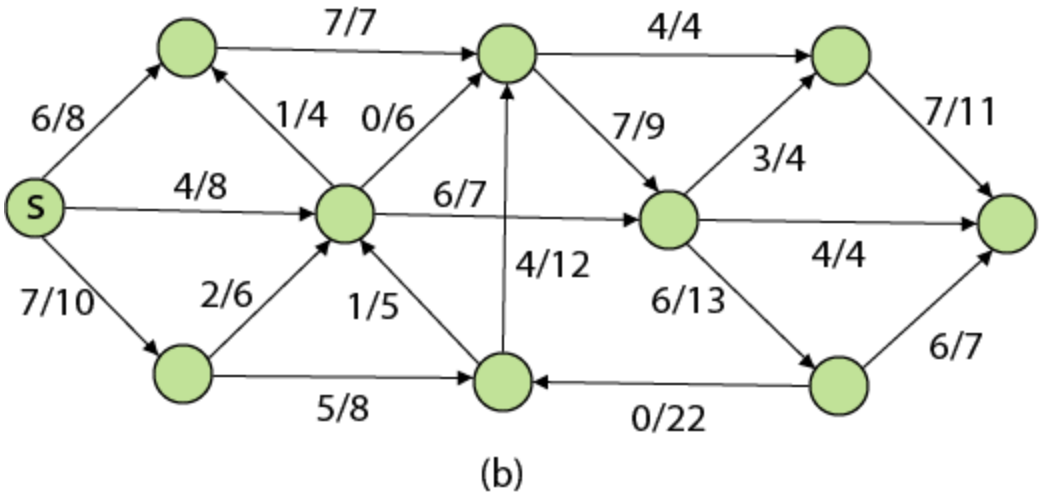
The quantity f (u, v), which can be positive or negative, is known as the net flow from vertex u to vertex v. In the **maximum-flow problem**, we are given a flow network G with source s and sink t, and we wish to find a flow of maximum value from s to t.

The three properties can be described as follows:

1. **Capacity Constraint** makes sure that the flow through each edge is not greater than the capacity.
2. **Skew Symmetry** means that the flow from u to v is the negative of the flow from v to u.
3. The flow-conservation property says that the total net flow out of a vertex other than the source or sink is 0. In other words, the amount of flow into a v is the same as the amount of flow out of v for every vertex v ∈ V - {s, t}



(a)



(b)

The value of the flow is the net flow from the source,

$$|f| = \sum_{v \in V} f(s, v)$$

The **positive net flow entering** a vertex v is described by

$$\sum_{\{u \in V : f(u,v) > 0\}} f(u, v)$$

The **positive net flow** leaving a vertex is described symmetrically. One interpretation of the Flow-Conservation Property is that the positive net flow entering a vertex other than the source or sink must equal the positive net flow leaving the vertex.

A flow f is said to be **integer-valued** if f (u, v) is an integer for all (u, v) ∈ E. Clearly, the value of the flow is an integer is an integer-valued flow.

TOPOLOGICAL SORTING

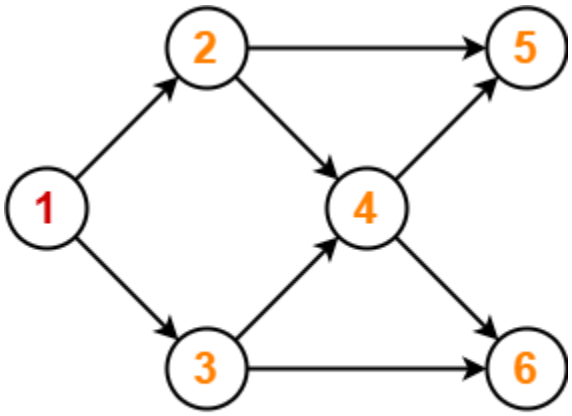
Topological Sort is a linear ordering of the vertices in such a way that if there is an edge in the DAG going from vertex 'u' to vertex 'v', then 'u' comes before 'v' in the ordering.

It is important to note that-

- Topological Sorting is possible if and only if the graph is a Directed Acyclic Graph.
- There may exist multiple different topological orderings for a given directed acyclic graph.

Topological Sort Example-

Consider the following directed acyclic graph-



Topological Sort Example

For this graph, following 4 different topological orderings are possible-

- 1 2 3 4 5 6
- 1 2 3 4 6 5
- 1 3 2 4 5 6

- 1 3 2 4 6 5

Applications of Topological Sort-

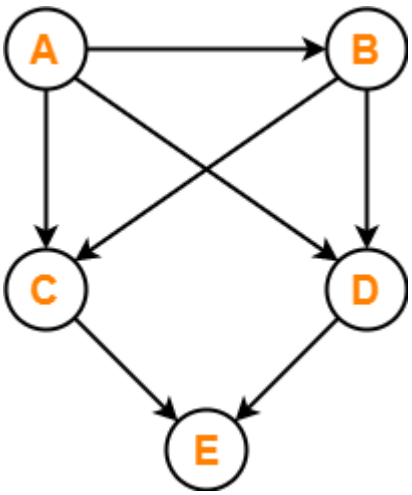
Few important applications of topological sort are-

- Scheduling jobs from the given dependencies among jobs
- Instruction Scheduling
- Determining the order of compilation tasks to perform in makefiles
- Data Serialization

PRACTICE PROBLEMS BASED ON TOPOLOGICAL SORT-

Problem-01:

Find the number of different topological orderings possible for the given graph-

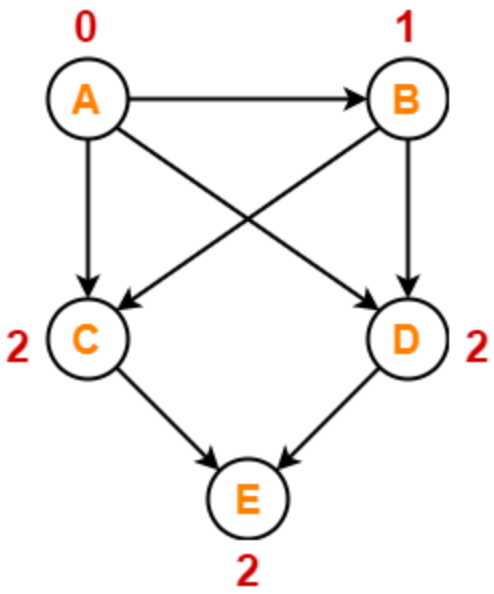


Solution-

The topological orderings of the above graph are found in the following steps-

Step-01:

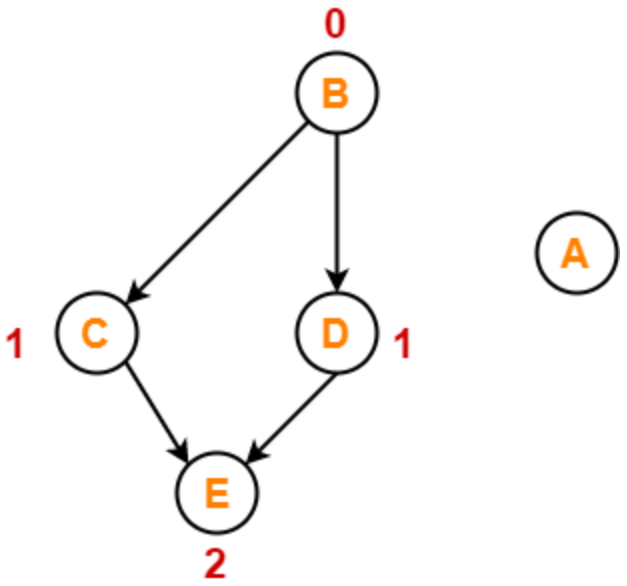
Write in-degree of each vertex -



Step-02:

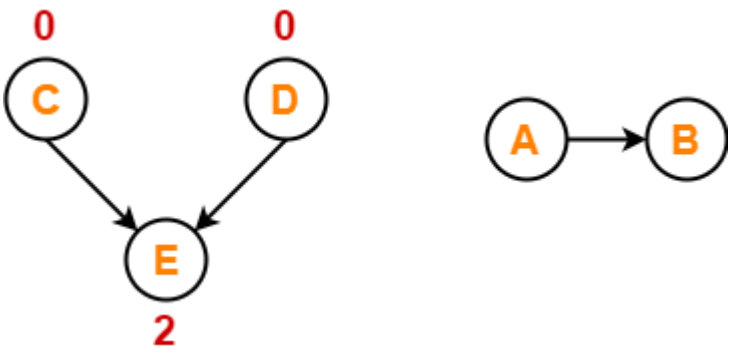
- Vertex-A has the least in-degree.
- So, remove vertex-A and its associated edges.

Now, update the in-degree of other vertices.



Step-03:

- Vertex-B has the least in-degree.
- So, remove vertex-B and its associated edges.
- Now, update the in-degree of other vertices.



Step-04:

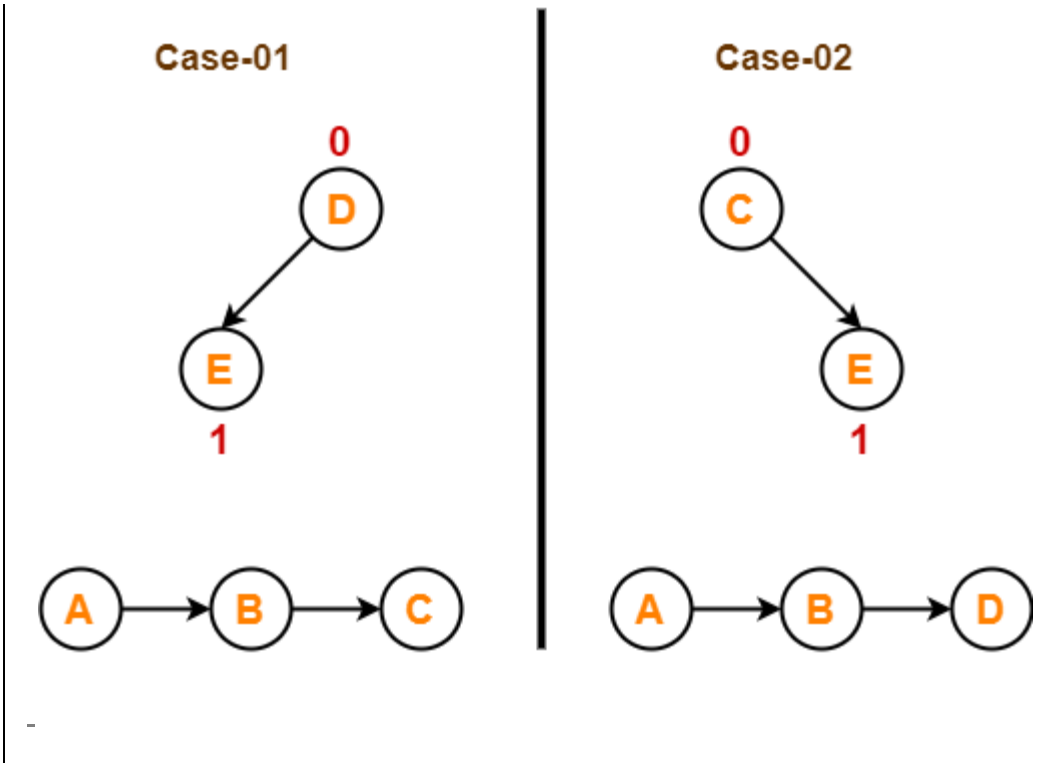
There are two vertices with the least in-degree. So, following 2 cases are possible-

In case-01,

- Remove vertex-C and its associated edges.
- Then, update the in-degree of other vertices.

In case-02,

- Remove vertex-D and its associated edges.
- Then, update the in-degree of other vertices.



Step-05:

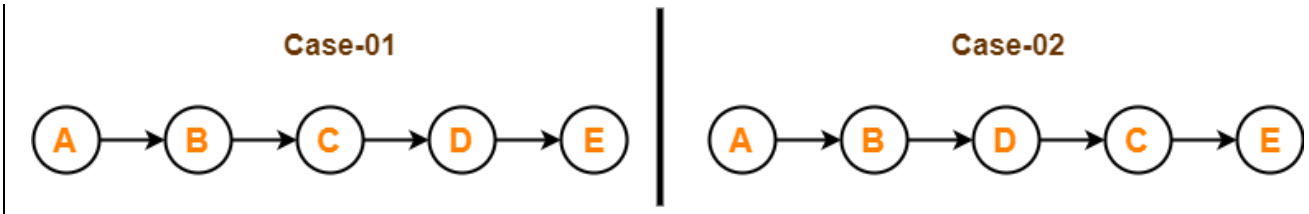
Now, the above two cases are continued separately in the similar manner.

In case-01,

- Remove vertex-D since it has the least in-degree.
- Then, remove the remaining vertex-E.

In case-02,

- Remove vertex-C since it has the least in-degree.
- Then, remove the remaining vertex-E.



Conclusion-

For the given graph, following 2 different topological orderings are possible-

- A B C D E
- A B D C E